Recitation 4

September 17

Review

A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if for any vector $b \in \mathbb{R}^m$ there is $x \in \mathbb{R}^n$ which being mapped to b, i.e. $T(x) = b$. If T is given by a matrix A, sending $x \mapsto Ax$, then T is onto $\Leftrightarrow$ system $Ax = b$ has a solution (is consistent) for every $b \Leftrightarrow$ every row of A is pivotal.

A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if for any vector $x \in \mathbb{R}^n$ $T(x) = 0$ implies $x = 0$, i.e. "T doesn't kill any non-zero vectors". If T is given by a matrix A, sending $x \mapsto Ax$, then T is one-to-one $\Leftrightarrow$ system $Ax = 0$ has only trivial solution $x = 0 \Leftrightarrow$ every column of A is pivotal $\Leftrightarrow$ columns of A are independent.

If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, to find its matrix A you need to see what T does to standard basis vectors $e_1, \dots, e_n$. Then $A = [T(e_1) \dots T(e_n)]$. So take e_1, apply T, it gives a vector in $\mathbb{R}^m$, that's your first column, etc.c.

Properties of inverse of a (square) matrix: $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$.

To find inverse of an $n \times n$ matrix A, write augmented matrix $[A|I_n]$, row reduce to get $[I_n|B]$. This B is A^{-1}.

Problems

Problem 1. Let $A = \begin{bmatrix} -2 & 3 & 4 & -5 \\ 4 & 2 & 1 & 1 \end{bmatrix}$. Do columns of A span $\mathbb{R}^2$? Do columns of A^T span $\mathbb{R}^4$?

Are columns of A linearly independent? What about A^T? Suppose equation $A^T x = b$ has at least one solution (b is some fixed vector in $\mathbb{R}^4$). How many solutions should it then have? To answer the last three questions use **pivots**.

Problem 2. For each of the following matrices, determine dimensions of domain and codomain of corresponding transformations, and determine if each of the transformations is **onto**, **one-to-one**, or both.

- $\begin{bmatrix} -1 & 3 & 4 \\ 3 & -4 & -2 \\ 0 & -3 & -6 \end{bmatrix}$
- $\begin{bmatrix} 4 & -1 & 3 \\ 4 & -4 & -2 \end{bmatrix}$

Problem 3. Let S be the transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ which reflects vectors first with respect to y-axis, then with respect to x-axis. Prove **using the definition of a linear transformation** that S is indeed linear.

Find the standard matrix of S.

Problem 4. Define a transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ which sends each vector $v \in \mathbb{R}^3$ to its mirror image with respect to the xy-plane. After that, apply transformation S sending a vector $[x_1, x_2, x_3]$ to $[3x_2 - x_1, x_2]$. Find standard matrices of T and S, and find the standard matrix of a transformation F which first applies T, then applies S.

Is F onto? Is it one-to-one? What about T and S?

Problem 5. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Verify that $AB = AC$, even though $B \neq C$. So the "cancellation law" doesn't work for matrices.

Is A invertible? Why?

Problem 6. Suppose for some matrices X, Y, Z you know that $XY = XZ$, and you know that X **is invertible**. What can you say about Y and Z?

Problem 7. Suppose $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and you know that for some matrix B, $AB = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \end{bmatrix}$. Find the matrix B.

Problem 8. Find the inverse of the matrix $C = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$

Problem 9. Suppose A, B, C are invertible $n \times n$ matrices. Does the equation $C^{-1}(A+X)B^{-1} = ABC$ have a solution, X? If yes, find it. What happens if A is not invertible?